GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

MATHEMATICS

B.Sc. Sem-1

PAPER-1:ALGEBRA AND TRIGONOMETRY

UNIT – IV

GROUP THEORY

SUBJECT: GROUP AND GENERAL PROPERTIES OF GROUP

❖ BINARY OPERATION :

Binary operation is a rule for combining two values to create a new value like addition , substraction ,multiplication and division on various sets of numbers.

In Mathematics , a binary operation on a set is a calculation that combines two elements of the set to produce another element of the set i.e. in binary operation domains and co-domains are the same set.

Let S be non-empty set .Then

$S \times S = \{(a, b) | a \in S, b \in S\}$ is a Cartesian product of S with itself.

Let '+' be the operation on the set thus '+' is called as the binary operation on the set S if and only if a + b ∈ S , for all a , b ∈ S and a + b is unique.

This is called closure property.

❖ ALGEBRAIC STRUCTURE :

A non empty set together with one or more binary operations defined on it and satisfying certain laws of binary operation is called as algebraic structure or algebraic system.

❖ GROUP :

A non empty set G is said to be a group , if in G there is defined a binary operation $*$ which satisfying following conditions :

1.Closure property :

$a , b \in G \Rightarrow a * b \in G$

2.Associative law :

$a , b , c \in G \Rightarrow a * (b * c) =(a * b) * c$

3.Existence of identity element in G:

$\exists\, e \in G$ such that $a * e = e * a = a , \forall\, a \in G$.

The element e is called the identity element of G.

4.Existence of inverse element in G:

For each $a \in G$, $\exists\, b \in G$ such that $a * b =b * a =e$

The element b is called the inverse of element a with respect to $*$.

❖ ABELIAN GROUP :

A group G is said to be abelian group (or commutative) group if $a \cdot b = b \cdot a, \forall a, b \in G$.

The group which is not abelian is called as non-abelian group.

The concept of abelian group is given by the Norwegian Mathematician Niels Henrik Abel (1802 -1829).

➢ NOTE:

1. The group with the binary operation ‘addition’ is known as additive group and that with multiplication is known as multiplicative group.

1. a)For addition operation ‘+’ , the identity element is 0. i.e. $a+0 = 0+a = a$ hence 0 is called as additive identity.
   b)For multiplication operation ‘ · ’ ,the identity element is 1.i.e. $a \cdot 1 = 1 \cdot a = a$ hence 1 is called as multiplicative identity.

3.a)For addition operation ‘ + ‘ , the inverse of a ∈ G is –a.

i.e. a + (-a) = 0 , a , -a ∈ G .

-a is called as additive inverse of a.

b)For multiplication operation ‘· ‘ , the inverse of a ∈ G is a^{-1}.

i.e. a · a^{-1} = a^{-1}· a =1

a^{-1} is called as multiplicative inverse of a.

4.If a group contains a finite number of element then it is called as finite group.

5.If a group contains infinite number of elements then it is called as infinite group.

6.The number of elements in a finite group G is called the order of a group G. It is denoted by O(G).

Example 1: Verify that the set Z of all integers is a infinite abelian group with respect to the operation of addition but not a group with respect to multiplication.

Solution : Let Z be the set of integers.

i.e. $Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

<u>Claim</u> : (Z, +) is an infinite abelian group.

1) <u>Closure property :-</u>

Let $x, y \in Z$ so that x and y are integers.

The sum of two integers is an integer and hence x+y is an integer.

i.e. $x, y \in Z \Rightarrow x+y \in Z$

$\Rightarrow$Z is closed w.r.t. addition.

2) <u>Associativity :-</u>

Since addition of integers are associative, associativity holds in Z.

i.e. $x + (y + z) = (x + y) + z, \forall x, y, z \in Z$

3) <u>Existence of identity element :</u>

$\exists\ 0 \in Z$ such that $x + 0 = 0 + x = x$, $x \in Z$

Hence 0 is an identity element w .r .t. '+' in Z.

4) <u>Existence of inverse element :</u>

For $a \in Z$, $\exists\ -a \in Z$ such that $a+(-a) = (-a)+a = 0$

$\Rightarrow$ -a is an inverse of $a \in Z$ w .r .t . addition.

$\therefore$ Every $a \in Z$ has an inverse in Z.

Thus all the conditions of groups are satisfied in Z w.r.t. addition.

$\therefore$ (Z,+) is a group.

Also, addition of integers are commutative.

i.e. $a+b = b+a$, $\forall\ a, b \in Z$

$\therefore$ (Z,+) is an abelian group.

Also, Z is infinite and hence (Z,+) is an infinite abelian group.

To prove : Z is not a group w.r.t. multiplication.
For any element a in Z except 1 and -1 , here does not exist an element b in Z such that a·b =b·a =1, an identity w.r.t. multiplication.
e.g:. For 2 ∈ Z ,$\nexists \frac{1}{2} \in$ Z s.t. $2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2$=1.
∴ Z is not a group w.r.t. multiplication.

❖ MODULO SYSTEM :

➢ Addition modulo m :-

The operation addition modulo m is denoted by "$+_m$" *and* defined as follows :
Let a and b be any two integers and m is fixed positive integer then a $+_m$ b = r , $0 \leq r \leq m$
where , r is the least non–negative remainder when a+b is divided by m.

- **Multiplication modulo m :-**

The operation multiplication modulo m is denoted by "$\cdot_m$"
Let a and b be any two integers and m is a fixed positive integer then a $\cdot_m$ b = r , $0 \leq r \leq m$
where , r is the least non–negative remainder when a·b is divided by m.

- **Congruence modulo m :-**

Let m be a fixed positive integer . If a and b are any two integers , we define a$\equiv b(mod\ m)$ if m|a-b (m divides a-b)
This relation is referred as congruence modulo m , m is called as modulus of the relation and a$\equiv b(mod\ m)$ read as a is congruent to b modulo m.

- **Modulo m set :**

The set of integers mod m is a finite set containing m elements 0,1,2,3,…,m-1.

❖ COMPOSITION TABLE :

A binary operation in a finite set can be completely described by a table. This table is called as composition table.

➢ Construction of composition table :-

1)Write the elements of the set in the row (top border row) as well as in column (extreme left column).

2)Enter the given operation at the top left blank corner.

3)Perform the row by column wise operation on the elements of the set and write down the outcome at the intersection of the row and the column.

For example,consider G={x , y , z} and the binary operation $*$. Then the composition table is given below:-

*	x	y	z
x	x*x	x*y	x*z
y	y*x	y*y	y*z
z	z*x	z*y	z*z

The composition tables are useful to examine the conditions of group as follows :

1) <u>Closure property :</u>

If all the entries in the table are in a set G then G is closed under the given operation. If any of the elements of the table does not belong to the set G then G is not closed.

2) <u>Existence of identity element :</u>

If there is some row R_i identical to the top (border) row in the table then the identit element exists and it is an element in the intersection of row R_i and extreme left (border) column.

3)<u>Existence of inverse element :</u>

Mark the identity element in the table. If the identity element is in the ith row and jth column then ith element in the extreme left (border) column is the inverse of the jth elements in the top (border) row and vice versa.

4)<u>Commutativity :</u>

If the composition table is symmetric about the principal diagonal then the composition is said to be commutative otherwise it is not commutative.

Question 1 :-If G consists of the two real numbers 1 and -1. Show that it is an abelian group of order 2 under the multiplication of two numbers.

Solution :- Let G={ 1 , -1 } be a finite set.

Here we construct the composition table for the multiplication in G as given below :

.	1	-1
1	1	-1
-1	-1	1

1) <u>Closure property :-</u>

Since all the entries in the composition table are elements of G the set G is closed under the operation multiplication.

2) <u>Associativity :-</u>

The multiplication of real numbers is associative and 1 and -1 are real numbers.

Hence, associativity holds in G.

3) <u>Existence of identity element in G :-</u>

Row 1 of the table is identical to the top (border) row and hence the element 1 in the extreme left (border) column heading row 1 is the identity element.

⇒Existence of identity under multiplication in G.

4) <u>Existence of inverse element in G :-</u>

From the table $1 \cdot 1 = 1$, $(-1) \cdot (-1) = 1$

$\Rightarrow$ 1 and -1 are their own inverses.

Thus every member of G has an inverse in G.

All the conditions of group are satisfied.

Hence G is a group.

Since the composition table is symmetric about the main diagonal, the commutative property w.r.t. multiplication holds in G. Also the no. of elements in G is 2(finite).

Hence G is a finite abelian group of order 2.