Govt. Institute of Science Ngpur

Power Point Presentation for

M. Sc. Semester I

Subject- Abstract Algebra

Conjugacy and G-set

Definition:- Let G be a group and X be a set. Then G is said to act on X if there is a mapping ø:G x X—>X, with ø(a, x)=a*x, such that for all a,b ϵ G, x ϵ X,

(i) a*(b*x)=(ab)*x,

(ii) e*x)=x.

The mapping ø is called the action of G on X and X is said to be a G-set.

Examples of G-sets

1) Let G be the additive group R, and X be the set of complex numbers z such that $|z| = 1$. Then X is a G-set under the action $a*x=\exp(ia)\, z$, where $a \in G$ and $x \in X$. Here the action of a is the rotation through an angle $\theta=a$ radians in anticlockwise.

2) Let G = S_5 and X = {x1,x2,x3,x4,x5} be a set of beads forming a circular ring. Then X is a G-set under the action σ*xj—xσ(j), σ ϵ G.

3) Let G = D4 and X be the vertices 1,2,3,4 of a square. X is a G-set under the action g*i=g(i), g ϵ D4, i ϵ {l,2,3,4).

4) Let G be a group. Define a*x=ax, a ϵ G, x ϵ G. Then, clearly, the set G is a G-set. This action of the group G on itself is called translation.

5) Let G be a group. Define $a*x=axa^{-1}$, a ϵ G, x ϵ G. We show that G is a G-set. Let a,b ϵ G. Then (ab)*x = $ab*x=abx(ab)^{-1}=a\left(bxb^{-1}\right)a^{-1}=a*(b*x)$.
Also, e * x = x. This proves G is a G-set. This action of the group G on itself is called conjugation.

Theorem:- Let G be a group and let X be a set.

(i) If X is a G-set, then the action of G on X induces a homomorphism

$\phi: G \longrightarrow S_x$,

(ii) Any homomorphism $\phi: G \longrightarrow S_x$, induces an action of G onto X.

Proof :-

(I) We are given that X is a G-set, hence G acts on X.

Let * be the action of G on X.

We define $\phi: G \longrightarrow S_x$ by $\phi(a)=f_a$ for all $a \in G$

where $f_a: X \longrightarrow X$ is a mapping defined as $f_a(x) = a*x$, $a \in G$, $x \in X$.

Clearly $f_a \in S_x$, for all $a \in G$.

Let a, $b \in G$.

Then $f_{ab}(x) = (ab)*x = a*(b*x) = f_a(f_b(x)) = f_af_b(x)$ for all $x \in X$ imply $\phi(ab) = f_{ab}$ $= f_af_b = \phi(a)\ \phi(b)$ i.e. ϕ is a group homomorphism.

(II) Let $\phi:G \longrightarrow S_X$ be a group homomorphism. Then $\phi(ab) = \phi(a)\,\phi(b)$.

Let $\phi(a)=f_a$, $a \in G$, where $f_a:X \longrightarrow X$ is a mapping defined as $f_a(x) = a*x$, $a \in G$, $x \in X$. $\Rightarrow f_{ab}=f_a f_b$

Then for $a, b \in G$, $(ab)*x=f_{ab}(x)=f_a f_b(x)=a*(b*x)$.

Also, $e*x = f_e(x) = x$, since $\phi(e)=f_e$ is an identity element of S_X.

Therefore by definition X is a G-set.