Govt. Institute of Science Ngpur

Power Point Presentation for

M. Sc. Semester III

Subject- General Relativity

A theory of gravity is a "metric theory" if and only if it can be given a mathematical representation in which two conditions hold:

Condition 1: There exists a symmetric metric tensor $g_{\mu\nu}$ of signature (−, +, +, +), which governs proper-length and proper-time measurements in the usual manner of special and general relativity:

$$d\tau^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu$$

where there is a summation over indices μ and ν

Condition 2: Stressed matter and fields being acted upon by gravity respond in accordance with the equation:

$$0 = \nabla_\nu T^{\mu\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu} T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\nu} T^{\mu\sigma}$$

where $T^{\mu\nu}$ is the stress–energy tensor for all matter and non-gravitational fields, and where ∇_ν is the covariant derivative with respect to the metric and $\Gamma^{\alpha}_{\sigma\nu}$ is the Christoffel symbol. The stress–energy tensor should also satisfy an energy condition.

Metric theories include (from simplest to most complex):

1) Scalar field theories (includes Conformally flat theories & Stratified theories with conformally flat space slices)

- Bergman
- Coleman
- Einstein (1912)
- Einstein–Fokker theory
- Lee- Lightman –NI
- Littlewood
- Ni
- Nordstromm's theory of gravitation (first metric theory of gravity to be developed)
- Page–Tupper
- Papapetrou
- Rosen (1971)
- Whitrow–Morduch
- Yilmaz theory of gravitation (attempted to eliminate event horizons from the theory)
- Quasilinear theories (includes Linear fixed gauge)
- Bollini–Giambiagi–Tiomno
- Deser–Laurent
- Whitehead's theory of gravity (intended to use only retarded potentials)

4) Vector-tensor theories

- Hellings–Nordtvedt
- Will–Nordtvedt
- Bimetric theories
- Lightman–Lee
- Rastall
- Rosen (1975)

  - Non-metric theories include

- Belinfante–Swihart
- Einstein–Cartan theory (intended to handle spin-orbital angular momentum interchange)
- Kustaanheimo (1967)
- Teleparallelism
- Gauge theory gravity

2) Tensor theories

- Einstein's GR
- Fourth order gravity (allows the Lagrangian to depend on second-order contractions of the Riemann curvature tensor)
- f(R) gravity (allows the Lagrangian to depend on higher powers of the Ricci scalar)
- Gauss-Bonnet gravity
- Lovelock theory of gravity (allows the Lagrangian to depend on higher-order contractions of the Riemann curvature tensor)

3) Scalar-tensor theories

- Bekenstein
- Bergmann-Wagoner
- Brans-Dicke theory (the most well-known alternative to GR, intended to be better at applying Mach's principle)
- Jordan
- Nordtvedt
- Thiry
- Chameleon
- Pressuron

- Hellings–Nordtvedt
- Will-Nordtvedt
- Bimetric theories
- Lightman-Lee
- Rastall
- Rosen (1975)

5) Non-metric theories

- Belinfante–Swihart
- Einsstein-Cartan theory (intended to handle spin-orbital angular momentum interchange)
- Kustaanheimo (1967)
- Teleparallelism
- Guage theory gravity

In Whitrow and Morduch (1960),

Scalar field theories

According to Page and Tupper (1968), who discuss all these except Nordström (1913), the general scalar field theory comes from the principle of least action:

$$\delta \int f\left(\frac{\phi}{c^2}\right) ds = 0$$

where the scalar field is,

$$\phi = GM/r$$

and *c* may or may not depend on ϕ

In Nordström (1912),

$$f(\phi/c^2) = \exp(-\phi/c^2), \qquad c = c_\infty$$

In Littlewood (1953) and Bergmann (1956),

$$f(\phi/c^2) = \exp(-\phi/c^2 - (\phi/c^2)^2/2), \qquad c = c_\infty$$

In Whitrow and Morduch (1960),

$$f(\phi/c^2) = 1, \qquad c^2 = c_\infty^2 - 2\phi$$

In Whitrow and Morduch (1965),

$$f(\phi/c^2) = \exp(-\phi/c^2), \qquad c^2 = c_\infty^2 - 2\phi$$

In Page and Tupper (1968),

$$f(\phi/c^2) = \phi/c^2 + \alpha(\phi/c^2)^2, \qquad c_\infty^2/c^2 = 1 + 4(\phi/c_\infty^2) + (15 + 2\alpha)(\phi/c_\infty^2)^2$$

Page and Tupper (1968) matches Yilmaz (1958) to second order when $\alpha = -7/2$

The gravitational deflection of light has to be zero when c is constant. Given that variable c and zero deflection of light are both in conflict with experiment, the prospect for a successful scalar theory of gravity looks very unlikely. Further, if the parameters of a scalar theory are adjusted so that the deflection of light is correct then the gravitational red shift is likely to be wrong.

Bimetric theories

Bimetric theories contain both the normal tensor metric and the Minkowski metric (or a metric of constant curvature), and may contain other scalar or vector fields. Rosen (1973, 1975) Bimetric Theory The action is:

$$S = \frac{1}{64\pi G}\int d^4x\sqrt{-\eta}\eta^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}(g_{\alpha\gamma|\mu}g_{\alpha\delta|\nu} - \tfrac{1}{2}g_{\alpha\beta|\mu}g_{\gamma\delta|\nu}) + S_m$$

where the vertical line "|" denotes covariant derivative with respect to . The field equations may be written in the form:

$$\Box_\eta g_{\mu\nu} - g^{\alpha\beta}\eta^{\gamma\delta}g_{\mu\alpha|\gamma}g_{\nu\beta|\delta} = -16\pi G\sqrt{g/\eta}(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)$$

Lightman-Lee (1973) developed a metric theory based on the non-metric theory of Belinfante and Swihart (1957a, 1957b). The result is known as BSLL theory. Given a tensor field $B_{\mu\nu}$, $B = B_{\mu\nu}\eta^{\mu\nu}$ and two constants a and f the action is:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\eta}(aB^{\mu\nu|\alpha}B_{\mu\nu|\alpha} + fB_{,\alpha}B^{,\alpha}) + S_m$$

and the stress–energy tensor comes from:

$$a\Box_\eta B^{\mu\nu} + f\eta^{\mu\nu}\Box_\eta B = -4\pi G\sqrt{g/\eta}T^{\alpha\beta}(\partial g_{\alpha\beta}/\partial B_\mu\nu)$$

In Rastall (1979), the metric is an algebraic function of the Minkowski metric and a Vector field.[5] The Action is:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}F(N)K^{\mu;\nu}K_{\mu;\nu} + S_m$$

Where $F(N) = -N/(2+N)$ and $N = g^{\mu\nu}K_\mu K_\nu$
for the field equation for $T^{\mu\nu}$ and K_μ

Quasilinear theories

In Whitehead (1922), the physical metric g is constructed (by Synge) algebraically from the Minkowski metric η and matter variables, so it doesn't even have a scalar field. The construction is:

$$g_{\mu\nu}(x^{\alpha}) = \eta_{\mu\nu} - 2\int_{\Sigma^{-}} \frac{y_{\mu}^{-} y_{\nu}^{-}}{(w^{-})^{3}} [\sqrt{-g}\rho u^{\alpha} d\Sigma_{\alpha}]^{-}$$

where the superscript (-) indicates quantities evaluated along the past η ,light cone of the field point x^{α}

$$(y^{\mu})^{-} = x^{\mu} - (x^{\mu})^{-}(y^{\mu})^{-}(y_{\mu})^{-} = 0,$$

$$w^{-} = (y^{\mu})^{-}(u_{\mu})^{-}\ (u_{\mu}) = dx^{\mu}/d\sigma,$$

$$d\sigma^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

Nevertheless the metric construction (from a non-metric theory) using the "length contraction" ansatz is criticized.
Deser and Laurent (1968) and Bollini-Giambiagi-Tiomno (1970) are Linear Fixed Gauge (LFG) theories. Taking an approach from quantum field theory, combine a Minkowski spacetime with the gauge invariant action of a spin-two tensor field (i.e. graviton) $h_{\mu\nu}$ to define

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The action is: $$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\eta}[2h^{\mu\nu}_{|\nu}h^{|\lambda}_{\mu\lambda} - 2h^{\mu\nu}_{|\nu}h^{\lambda}_{\lambda|\mu} + h^{\nu}_{\nu|\mu}h^{\lambda|\mu}_{\lambda} - h^{\mu\nu|\lambda}h_{\mu\nu|\lambda}] + S_m$$

Tensor theories

Einstein's general relativity is the simplest plausible theory of gravity that can be based on just one symmetric tensor field (the metric tensor). Others include: Gauss–Bonnet gravity, f(R) gravity, and Lovelock theory of gravity.

Scalar-tensor theories

Scalar-Tensor theories include Thiry (1948), Jordan (1955), Brans and Dicke (1961), Bergman (1968), Nordtveldt (1970), Wagoner (1970), Bekenstein (1977) and Barker (1978).

The action S is based on the integral of the Lagrangian L_ϕ

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}L_\phi + S_m$$

$$L_\phi = \phi R - \frac{\omega(\phi)}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2\phi\lambda(\phi)$$

$$S_m = \int d^4x\sqrt{g}G_N L_m$$

$$T^{\mu\nu} \stackrel{\text{def}}{=} \frac{2}{\sqrt{g}}\frac{\delta S_m}{\delta g_{\mu\nu}}$$

Where $\omega(\phi)$ is a different dimensionless function for each different scalar-tensor theory. The function $\lambda(\phi)$ plays the same role as the cosmological constant in GR. G_N is a dimensionless normalization constant that fixes the present-day value of G An arbitrary potential can be added for the scalar. The full version is retained in Bergman (1968) and Wagoner (1970). Special cases are:

Nordtvedt (1970), $\lambda = 0$

Vector-tensor theories

Hellings and Nordtvedt (1973) and Will and Nordtvedt (1972) are both vector-tensor theories. In addition to the metric tensor there is a timelike vector field K_μ

The gravitational action is:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R+\omega K_\mu K^\mu R+\eta K^\mu K^\nu R_{\mu\nu}-\epsilon F_{\mu\nu}F^{\mu\nu}+\tau K_{\mu;\nu}K^{\mu;\nu}]+S_m$$

Where ω, η, ϵ and τ are constants and

$$F_{\mu\nu} = K_{\nu;\mu} - K_{\mu;\nu}$$

for the field equations for $T^{\mu\nu}$ and K_μ.

Will and Nordtvedt (1972) is a special case where

$$\omega = \eta = \epsilon = 0, \tau = 1$$

Hellings and Nordtvedt (1973) is a special case where

$$\tau = 0 ; \epsilon = 1 ; \eta = -2\omega$$

These vector-tensor theories are semi-conservative, which means that they satisfy the laws of conservation of momentum and angular momentum but can have preferred frame effects. When $\omega = \eta = \epsilon = \tau = 0$ they reduce to GR so, so long as GR is confirmed by experiment, general vector-tensor theories can never be ruled out.

Other metric theories

Others metric theories have been proposed; that of Bekenstein (2004) is discussed under Modern Theories.

Since λ was thought to be zero at the time anyway, this would not have been considered a significant difference. The role of the cosmological constant in more modern work is discussed under Cosmological constant.

Brans–Dicke (1961), ω is constant

Bekenstein (1977) Variable Mass Theory Starting with parameters r and q found from a cosmological solution, $\phi = [1 - qf(\phi)]f(\phi)^{-r}$, determines function f then

$$\omega(\phi) = -\tfrac{3}{2} - \tfrac{1}{4} f(\phi)[(1-6q)qf(\phi) - 1][r + (1-r)qf(\phi)]^{-2}$$

Barker (1978) Constant G Theory

$$\omega(\phi) = (4 - 3\phi)/(2\phi - 2)$$

Adjustment of $\omega(\phi)$ allows Scalar Tensor Theories to tend to GR in the limit of $\omega \to \infty$ in the current epoch. However, there could be significant differences from GR in the early universe.

So long as GR is confirmed by experiment, general Scalar-Tensor theories (including Brans–Dicke) can never be ruled out entirely, but as experiments continue to confirm GR more precisely and the parameters have to be fine-tuned so that the predictions more closely match those of GR.

Non-metric theories

Cartan's theory is particularly interesting both because it is a non-metric theory and because it is so old. The status of Cartan's theory is uncertain. Will (1981) claims that all non-metric theories are eliminated by Einstein's Equivalence Principle (EEP). Will (2001) tempers that by explaining experimental criteria for testing non-metric theories against EEP. Misner et al. (1973) claims that Cartan's theory is the only non-metric theory to survive all experimental tests up to that date and Turyshev (2006) lists Cartan's theory among the few that have survived all experimental tests up to that date. The following is a quick sketch of Cartan's theory as restated by Trautman (1972).

Cartan (1922, 1923) suggested a simple generalization of Einstein's theory of gravitation. He proposed a model of space time with a metric tensor and a linear "connection" compatible with the metric but not necessarily symmetric. The torsion tensor of the connection is related to the density of intrinsic angular momentum. Independently of Cartan, similar ideas were put forward by Sciama, by Kibble in the years 1958 to 1966, culminating in a 1976 review by Hehl et al.

The original description is in terms of differential forms, but for the present article that is replaced by the more familiar language of tensors (risking loss of accuracy). As in GR, the Lagrangian is made up of a massless and a mass part. The Lagrangian for the massless part is:

$$L = \frac{1}{32\pi G}\Omega^{\mu}_{\nu} g^{\nu\xi} x^{\eta} x^{\zeta} \varepsilon_{\xi\mu\eta\zeta}$$

$$\Omega^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\eta}_{\xi}$$

$$\nabla x^{\mu} = -\omega^{\mu}_{\nu} x^{\nu}$$

The ω^{μ}_{ν} is the linear connection. $\varepsilon_{\xi\mu\eta\zeta}$ is the completely antisymmetric pseudo-tensor (Levi-Civita symbol) with $\varepsilon_{0123} = \sqrt{-g}$ and $g^{\nu\xi}$ is the metric tensor as usual. By assuming that the linear connection is metric, it is possible to remove the unwanted freedom inherent in the non-metric theory. The stress–energy tensor is calculated from:

$$T^{\mu\nu} = \frac{1}{16\pi G}(g^{\mu\nu}\eta^{\xi}_{\eta} - g^{\xi\mu}\eta^{\nu}_{\eta} - g^{\xi\nu}\eta^{\mu}_{\eta})\Omega^{\eta}_{\xi}$$

The space curvature is not Riemannian, but on a Riemannian space-time the Lagrangian would reduce to the Lagrangian of GR.

Testing of alternatives to general relativity:

- **Self-consistency**
- **Completeness**
- **Classical tests**
- **The Einstein equivalence principle (EEP)**
- **Parametric post-Newtonian (PPN) formalism**
- **Strong gravity and gravitational wave**
- **Cosmological tests**