M.Sc. Sem. I Mathematics

Paper IV

Linear Algebra

Uniform norm on $L(R^n)$

$If\ \mathrm{T}:\mathrm{R}^{\mathrm{n}} \to R^n$ is an operator, the uniform norm of T is defined to be

$$\|\mathrm{T}\| = \max\{|\mathrm{Tx}| / |x| \le 1\}.$$

Lemma1. *Let* R^{n} *be* given a norm $|\mathrm{x}|$. *The* corresponding uniform norm on $\mathrm{L}(\mathrm{R}^{\mathrm{n}})$ has the following properties:

(a) If $\|\mathrm{T}\| = k$, then $|\mathrm{Tx}| \le k|x|$ for all x in R^{n}.

$(b)\ \|\mathrm{ST}\| \le \|S\|.\|T\|.$

$(c)\ \|\mathrm{T}^{\mathrm{m}}\| \le \|T\|^m$ for all $\mathrm{m} = 0,1,2,\cdots$.

Proposition. Let P,S,T denote operators on R^n .Then:

(a) if $Q = PTP^{-1}$, *then* $e^Q = Pe^T P^{-1}$;

(b) if $ST = TS$, then $e^{S+T} = e^S e^T$;

(c) $e^{-S} = \left(e^S\right)^{-1}$;

(d) if $n = 2$ and $T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, *then*

$$e^T = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}.$$

The Primary Decomposition

Let E be a vector space real or complex and T be an operator on E. Assume that T has distinct real eigenvalues say $\lambda_1, \cdots, \lambda_k$.

The characteristic polynomial of T is given as

$$p(t) = \prod_{k=1}^{r} (t - \lambda_k)^{n_k}.$$

Here the integer n_k is the multiplicity of λ_k; note that

$$n_1 + \cdots + n_k = \dim E.$$

The generalized eigenspace of T belonging to λ_k is defined to be the subspace

$$E(T, \lambda_k) = Ker(T - \lambda_k I)^{n_k} \subset E.$$

This subspace is invariant under T.

Primary Decomposition Theorem: Let T be an operator on E, where E is real or complex vector space and T has real eigenvalues. Then E is the direct sum of the generalized eigenspaces of T. The dimension of each generalized eigenspace equals the multiplicity of the corresponding eigenvalue.

Theorem: Let $T \in L(E)$,where E is complex if T has a nonreal eigenvalue . Then $T=S+N$,where $SN=NS$ and S is diagonalizable and N is nilpotent.