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Class:- B.Sc.(sem-I)

Subject:-Mathematics

paper II:- Calculus

Topic:- convergent sequence in a metric space

convergent sequence in a metric space

Definition:-

A sequence $\{P_n\}$ *in a metic space* (X, d) *is said to be* $\boldsymbol{converge}$ *if there is a point* $p \epsilon X$ with the following property : $\forall\ \epsilon > 0, \exists$ *an integer* N *such that* $d(P_n, p) < \epsilon$, whenever $n \geq N$.

We also say that $\{P_n\}$ converges to p, or thst p is the limit of $\{P_n\}$, and we write $\lim_{n\to\infty} P_n = p$ or $P_n \to p$ *as* $n \to \infty$, *or simply* $P_n \to p$.

Definition:-divergence

Equivalently, we say that $\{ P_n \}$ converges to p, if for every open sphere $S_\in$(p) with a centre at P, $\exists$ $a\ positive\ integer\ N\ such\ that\ P_n \in S_\in$(p) , whenever n $\geq N$

It there is no such points p in X , the sequence $\{P_n\}$ is said to **diverge.**

<u>Open sphere:-</u>

Let (X, d) be a metric space and $p\epsilon$X . Let r be a positive real number, The open sphere with a centre p and radius r is a subset of X denoted and defined by

$$\boldsymbol{S_\in}\textbf{(p) = \{}\boldsymbol{q \in X/d(p, q) <}\textbf{r\}}$$

Complete metric space:-

A sequence $\{P_n\}$ ***in a metic space*** (X, d) ***is said to be cauchy sequence if*** $\forall\, \epsilon > 0$, $\exists$ ***an integer N such that*** $d(P_n, P_m) < \epsilon$, ***if*** $n \geq N$ ***and*** $m \geq N$

Ex:-If $\{P_n\} = \{\frac{1}{n}\}$, *then sequence* $\{P_n\}$ *is a Cauchy sequence in R.*

Solution:- We have $d(P_n, P_m) = |P_m - P_n|$ *in R, d(x,y)= |x-y|*

$= \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n}$ ....(1)

If m,n $\geq$ N , then $\frac{1}{m}, \frac{1}{n} \leq \frac{1}{N}$

from (1) and (2), m, $n \geq N \Rightarrow d(P_n, P_m) \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$

$\frac{2}{N} < \epsilon$, *we have* $d(P_n, P_m) < \epsilon$ *whenever* $n, m \geq N$.

$N > \frac{2}{\epsilon} \Rightarrow d(P_n, P_m) < \epsilon$ *whenever* $n, m \geq N$.

For given $\epsilon > 0$, $\exists$ *an integer N such that* $N > \frac{2}{\epsilon}$ *such that* $d(P_n, P_m) < \epsilon$ *whenever* $n, m \geq N$.

$\Rightarrow \{P_n\} = \{\frac{1}{n}\}$, *is a Cauchy sequence in R*

<u>Theorem</u> :-
Every convergent sequence in a metric space is a Cauchy sequence.

Proof:-

let $\{P_n\}$ be a convergent in a metric space (X, d) and it converges to $p \epsilon X$.

Thus,

$\forall\ \epsilon > 0, \exists\ an\ integer\ N\ such\ that\ d(P_n, p) < \frac{\epsilon}{2}$, whenever n $\geq N$.

Similarly, $d(P_m, p) < \frac{\epsilon}{2}$, whenever m $\geq N$.

Now, since d is a metric, we have

$d(P_n, P_m) \leq d(P_m, p) + d(p, P_m)$ (by triangle inequality)

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \in$ $\forall\ \epsilon > 0\ whenever\ n, m \geq N$

$\boldsymbol{\forall\ \epsilon > 0, \exists\ an\ integer\ N\ such\ that\ d(P_n, P_m) < \epsilon}$***, if n*** $\boldsymbol{\geq N}$ ***and m*** $\boldsymbol{\geq N}$

$\{P_n\}$ *is a Cauchy sequence.*

Every convergent sequence in a metric space is a Cauchy sequence.

Thank you