GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

MATHEMATICS

B.Sc. Sem-1

PAPER-1:ALGEBRA AND TRIGONOMETRY

UNIT – III

TRIGONOMETRY

SUBJECT: De Moivre's theorem and q-roots

❖ DE MOIVRE'S THEOREM:

Statement:

1.If n is a positive or negative integer then $(\cos\theta + isin\theta)^n = \cos n\theta + i\sin n\theta$

2.If n is a positive or negative fraction ,then $(\cos n\theta + i\sin n\theta)$ is one of the value of $(\cos\theta + i\sin\theta)^n$,where $\theta \epsilon R$.

Proof : To prove the De Moivre's theorem ,we consider the following three cases.

Case 1: When n is a positive integer.

Let be $(\cos\alpha + i\sin\alpha)$ and $(\cos\beta + i\sin\beta)$ two complex numbers.

By multiplication of complex numbers , we get

$$(\cos\alpha + i\sin\alpha).(\cos\beta + i\sin\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

Multiplying both sides by $(\cos\Upsilon + i\sin\Upsilon)$,we obtain

$$\text{Cis}(\alpha)\text{cis}(\beta)\text{cis}(\Upsilon) = [\text{cis}(\alpha+\beta)]\text{cis}(\Upsilon)$$
$$= \text{cis}(\alpha+\beta+\Upsilon)$$

Continuing in this way,

Cis (α) cis (β)cis (Υ) …upto n factors= cis $(\alpha+\beta+\Upsilon+\ldots$ upto n terms)

Putting α=β=ϒ=...=θ on both the sides, we get

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Thus,DMT is proved for positive integer n.

Case-2: When n is any negative integer.

Then, we can write n=-m,where m is positive integer.

Now,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{...by case-1}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \cdot \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$(\cos\theta + i\sin\theta)^n = \frac{\cos(-m)\theta + i\sin(-m)\theta}{1}$$

$$= \cos n\theta + i\sin n\theta$$

Hence, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ when n is negative.

$\Rightarrow$DMT is proved for negative integer n.

Case-3: When n is a fraction , positive or negative.

Then we can write $n = \frac{p}{q}$,where $p \in I$, $q \in N$.

As q is positive integer, by case-1 of DMT, we get

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q\,\frac{p}{q}\right) + i\sin\left(q\,\frac{p}{q}\right)$$

$$= \cos\theta + i\sin\theta$$

$$\therefore \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right) = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$\Rightarrow \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the value of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

Raising the power p to both the sides , we get

$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^{p}$ is one of the value of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$

$\Rightarrow\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the value of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$

i.e. $\cos n\theta+i\sin n\theta$ is one of the value of $(\cos\theta+i\sin\theta)^{n}$.

Thus , DMT is proved for any fraction n.

In this way , from the three cases above , De Moivre's Theorem follows.

- Example based on De Moivre's Theorem:

Question : Prove that:-

$$(a+ib)^{\frac{m}{n}}+(a-ib)^{\frac{m}{n}}=2(a^2+b^2)^{\frac{m}{2n}}.\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$$

Solution : Let a=r cosθ and b=r sinθ

Then,r=$\sqrt{a^2+b^2}$ and θ=$\tan^{-1}\frac{b}{a}$

∴ a+ib = rcosθ +i rsinθ

∴ a+ib =r[cosθ +i sin θ]

Raising power $\frac{m}{n}$ on both the sides , we get

$$(a+ib)^{\frac{m}{n}} = r^{\frac{m}{n}}[\cos\theta + i\sin\theta]^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left[\cos\left(\frac{m}{n}\right)\theta + i\sin\left(\frac{m}{n}\right)\theta\right] \qquad \text{…by DMT}$$

Similarly,

$$(a-ib)^{\frac{m}{n}} = r^{\frac{m}{n}}\left[\cos\left(\frac{m}{n}\right)\theta - i\sin\left(\frac{m}{n}\right)\theta\right]$$

So that $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2r^{\frac{m}{n}}\cos\frac{m}{n}\theta$

$$\therefore (a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2\left(\sqrt{a^2+b^2}\right)^{\frac{m}{n}}.\cos\frac{m}{n}\tan^{-1}\frac{b}{a}$$

$$= 2(a^2+b^2)^{\frac{m}{2n}}\cos\frac{m}{n}\tan^{-1}\frac{b}{a}$$

Hence proved.

❖ The q-roots of $(\cos\theta + \boldsymbol{i}\sin\boldsymbol{\theta})^{\frac{p}{q}}$:

By De Moivre's Theorem , if n is a positive or negative fraction then $\cos n\theta + i\sin n\theta$ is one of the value of $(\cos\theta + i\sin\theta)^n$,where $\theta \epsilon$R. But we should have n values of nth roots ,to be determined as follows.

Some standard forms of 1, -1, i, -i.

1$=\cos 0 + i\sin 0$

-1$=\cos\pi + i\sin\pi$

i$=\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

-i$=\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

Question :Solve the equation $x^7 + 1 = 0$ by De Moivre's theorem.

Solution :Given equation is $x^7 + 1 = 0$

$\Rightarrow x^7 = -1$

$\therefore x = (-1)^{\frac{1}{7}}$

Denote z = -1

Then z =$[\cos\pi + i\sin\pi]$

We know that, sine and cosine functions are periodic functions with period 2π.

$\therefore$ z=$[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]$, where k = 0,1,2,3,....

Raising power $\frac{1}{7}$ throughout, we get

$$z^{\frac{1}{7}} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{7}}$$

Say $u_k = z^{\frac{1}{7}} = [\cos(2k+1)\,{}^{\pi}/_{7} + i\sin(2k+1)\,{}^{\pi}/_{7}]$ ...eqn (1)

For k=0, (1)$\Rightarrow u_0 = [\cos(\pi/7) + i\sin(\pi/7)]$

For k=1, (1)$\Rightarrow u_1 = [\cos(3\pi/7) + i\sin(3\pi/7)]$

For k=2, (1)$\Rightarrow u_2 = [\cos(5\pi/7) + i\sin(5\pi/7)]$

For k=3, (1)$\Rightarrow u_3 = [\cos(\pi) + i\sin(\pi)]$=-1

For k=4, (1)$\Rightarrow u_4 = [\cos(9\pi/7) + i\sin(9\pi/7)]$

$$= [\cos(2\pi - 5\pi/7) + i\sin(2\pi - 5\pi/7)]$$

$$= [\cos(5\pi/7) - i\sin(5\pi/7)]$$

For k=5, (1)$\Rightarrow u_5 = [\cos(11\pi/7) + i\sin(11\pi/7)]$

$$= [\cos(2\pi - 3\pi/7) + i\sin(2\pi - 3\pi/7)]$$

$$=[\cos(3\pi/7) - i\sin(3\pi/7)]$$

For k=6, (1)⇒$u_6 = [\cos({}^{13\pi}/_7) + i\sin({}^{13\pi}/_7)]$

$= [\cos({}^{2\pi-\pi}/_7) + i\sin({}^{2\pi-\pi}/_7)]$

$=[\cos({}^{\pi}/_7) - i\sin({}^{\pi}/_7)]$

These are seven distinct roots of $(-1)^{\frac{1}{7}}$.

Thus the required roots of given equation are -1,$[\cos({}^{r\pi}/_7) \pm \sin({}^{r\pi}/_7)]$ where r=1,3,5.