GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

MATHEMATICS

B.Sc. Sem-1

PAPER-1:ALGEBRA AND TRIGONOMETRY

UNIT – III

TRIGONOMETRY

SUBJECT: Hyperbolic functions,inverse hyperbolic functions,logarithms of complex quantity and Gregory,s series

❖HYPERBOLIC FUNCTIONS:

If z is real or complex, then the quantity $\left(\frac{e^z+e^{-z}}{2}\right)$ is called hyperbolic cosine of z . It is denoted by cosh z (hyperbolic cosine function of z).

$$\cosh z = \left(\frac{e^z+e^{-z}}{2}\right)$$

The quantity $\left(\frac{e^z-e^{-z}}{2}\right)$ is called a hyperbolic sine function of z. It is denoted by sinh z (hyperbolic sine function of z).

$$\sinh z = \left(\frac{e^z-e^{-z}}{2}\right)$$

Analogically,other hyperbolic function are defined as follows:

$$\tanh z = \frac{e^z-e^{-z}}{e^z+e^{-z}}, \quad \coth z = \frac{e^z+e^{-z}}{e^z-e^{-z}}$$

$$\text{cosech}\, z = \frac{2}{e^z-e^{-z}}, \quad \text{sech}\, z = \frac{2}{e^z+e^{-z}}$$

➢ Relation between hyperbolic and circular function :

1. $\cos(iz) = \cosh z$
2. $\sin(iz) = i \sinh z$
3. $\tan(iz) = i \tanh z$
4. $\sec(iz) = \operatorname{sech} z$
5. $\operatorname{cosec}(iz) = -i \operatorname{cosech} z$
6. $\cot(iz) = -i \coth z$

Separation of complex function into real and imaginary parts :

Question-1: Separate into real and imaginary parts of $\sin(\alpha + i\beta)$

Solution :- Let z = $\sin(\alpha + i\beta)$

$$= \sin\alpha \cos(i\beta) + \cos\alpha \sin(i\beta)$$

$$= \sin\alpha \cosh\beta + i \cos\alpha \sinh\beta$$

Hence, Re(z) $= \sin\alpha \cosh\beta$

Im(z) $= \cos\alpha \sinh\beta$

Question-2: If $\cos(x + iy) = \cos\alpha + i\sin\alpha$, show that

$$\cosh 2y + \cos 2x = 2.$$

Solution :-Let $\cos(x + iy) = \cos\alpha + i\sin\alpha$

$$\Rightarrow \cos x \cos iy - \sin x \sin iy = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \cos x \cosh y - i\sin x \sinh y = \cos\alpha + i\sin\alpha$$

Equating real and imaginary parts, we get

$\cos x \cosh y = \cos \alpha$ ...(1)

$-\sin x \sinh y = \sin \alpha$ ...(2)

Squaring and adding equations (1) and (2), we get

$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 \alpha + \sin^2 \alpha$

$\therefore \cos^2 x \cosh^2 y + (1 - \cos^2 x)(\cosh^2 y - 1) = 1$

$\Rightarrow \cos^2 x \cosh^2 y + \cosh^2 y - 1 - \cos^2 x \cosh^2 y + \cos^2 x = 1$

$\Rightarrow \cosh^2 y + \cos^2 x = 2$

$\Rightarrow \frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2} = 2$

$\Rightarrow \frac{1}{2} + \frac{1}{2} + \frac{\cosh 2y}{2} + \frac{\cos 2x}{2} = 2$

$\Rightarrow \frac{1}{2}[\cosh 2y + \cos 2x] = 2 - 1 = 1$

$\Rightarrow \cosh 2y + \cos 2x = 2$

Hence proved.

❖ INVERSE HYPERBOLIC FUNCTIONS :

If sin z =u then z $=\sin^{-1} u$ is called inverse huperbolic sine function of u.

Similarly other inverse hyperbolic functions are defined.

The inverse hyperbolic function can be expressed as logarithmic functions as below :

1. If $\sinh y = \text{x}$ then $y = \sinh^{-1} x = \log\left[x + \sqrt{x^2 + 1}\right]$.
2. If $\cosh y = x \; then \; y = \cosh^{-1} x = \log\left[x + \sqrt{x^2 - 1}\right]$
3. If $\tanh y = x$ then $y = \tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

➢ Relation between inverse hyperbolic and inverse circular functions :

1. $\sinh^{-1} y = \left(\frac{1}{i}\right) \sin^{-1}(iy)$

2. $\cosh^{-1} y = \left(\frac{1}{i}\right) \cos^{-1}(y)$

3. $\tanh^{-1} y = \left(\frac{1}{i}\right) \tan^{-1}(iy)$

➢ Separation of complex function into real and imaginary parts :

Question-1:Separate into real and imaginary parts of $\tan^{-1}(\alpha + i\beta)$

Solution :-Let $\tan^{-1}(\alpha + i\beta)$=x+iy

$$\Rightarrow \alpha + i\beta = \tan(x + iy)$$

$$\Rightarrow \alpha - i\beta = \tan(x - iy)$$

We can write $2x = (x + iy) + (x - iy)$

Then $\tan 2x = \tan[(x + iy) + (x - iy)]$

$$\tan 2x = \frac{\tan(x + iy) + \tan(x - iy)}{1 - \tan(x + iy)\tan(x - iy)}$$

$$= \frac{(\alpha+i\beta)+(\alpha-i\beta)}{1-(\alpha+i\beta)(\alpha-i\beta)}$$

$$= \frac{2\alpha}{1-[\alpha^2-(i\beta)^2]}$$

$$= \frac{2\alpha}{1-[\alpha^2+\beta^2]}$$

$$\Rightarrow 2x = \tan^{-1}\left[\frac{2\alpha}{1-\alpha^2-\beta^2}\right]$$

$$\Rightarrow x = \frac{1}{2}\tan^{-1}\left[\frac{2\alpha}{1-\alpha^2-\beta^2}\right] = \mathrm{Re}\{\tan^{-1}(\alpha + i\beta)\}$$

Also we can write $2iy = (x + iy) - (x - iy)$

Then, $\tan(2iy) = \tan[(x + iy) - (x - iy)]$

$$\mathrm{i}\tanh 2y = \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\alpha+i\beta)-(\alpha-i\beta)}{1+(\alpha+i\beta)(\alpha-i\beta)}$$

$$\mathrm{i}\tanh 2y = \frac{2i\beta}{1+[\alpha^2+\beta^2]}$$

$$\tanh 2y = \frac{2\beta}{1+[\alpha^2+\beta^2]}$$

$$\therefore 2y = \tanh^{-1}\left(\frac{2\beta}{1+[\alpha^2+\beta^2]}\right)$$

$$\therefore y = \frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{1+[\alpha^2+\beta^2]}\right)$$

We know that $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$\therefore y = \frac{1}{2}\left\{\frac{1}{2}\log\left[\frac{1+\left(\frac{2\beta}{1+[\alpha^2+\beta^2]}\right)}{1-\left(\frac{2\beta}{1+[\alpha^2+\beta^2]}\right)}\right]\right\}$$

$$= \frac{1}{4}\log\left[\frac{1+\alpha^2+\beta^2+2\beta}{1+\alpha^2+\beta^2-2\beta}\right]$$

$$\therefore y = \frac{1}{4}\log\left[\frac{(1+\beta)^2+\alpha^2}{(1-\beta)^2+\alpha^2}\right] = \mathrm{Im}\{\tan^{-1}(\alpha + i\beta)\}$$

❖ Logarithms of complex quantities :

If $e^{x+iy} = u + iv$ then $x + iy = \log(u + iv)$ is called a logarithm of complex quantity u+iv.

By Euler's identity,

$$e^{2n\pi i} = \cos(2n\pi) + i\sin(2n\pi)$$

$$e^{2n\pi i} = 1, \text{ where } n \epsilon I$$

$\therefore e^{x+iy} e^{2n\pi i} = (u+iv).1$

$\therefore e^{x+i(y+2n\pi)} = u+iv$

$\Rightarrow x + i(y + 2n\pi) = \log_e(u + iv)\, , n\epsilon I$

This shows that the logarithm of the complex quantity is a many valued functions.

The general value of logarithm of u+iv is denoted by Log (u+iv) and the principal value is given by $\log_e(u + iv)$ for which n=0.

Hence,

$$\text{Log}\,(u+iv) = x+i(y+2n\pi)$$

$$= (x+iy) + i2n\pi$$

$$\text{Log}\,(u+iv) = \log\,(u+iv) + 2n\pi i$$

➢ Some special cases :

1. Logarithm of a negative real quantity :-

The principal value of a negative real quantity is given by,

$$\log(-x) = \log x + i\pi$$

The general value of a negative real quantity is given by,

$$\text{Log}(-x) = \log(x) + i(1+2n)\pi$$

2. Logarithm of a purely imaginary quantity:-

The principal value of a purely imaginary quantity is given by,

$$\log(xi) = \log x + i\frac{\pi}{2}$$

The general value of a purely imaginary quantity is given by,

$$\text{Log}(xi) = \log x + i\left(\frac{1}{2} + 2n\right)\pi$$

Question 1: Find general value of log (4+3i).

Solution : Let 4+3i $= re^{i\theta}$

where r $=\sqrt{4^2+3^2}=\sqrt{25}=5$ and $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

Now,

$$\log(4+3i) = \log(re^{i\theta})$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$

$$\therefore \log(4+3i) = \log 5 + i\tan^{-1}\left(\frac{3}{4}\right) \quad \ldots(1)$$

This is the principal value of log (4+3i).

The general value is given by,

$$\text{Log}(4+3i) = \log(4+3i) + 2n\pi i,\ n\epsilon I$$
$$= \log 5 + i\tan^{-1}\left(\frac{3}{4}\right) + 2n\pi i \quad \ldots\text{by (1)}$$

$$\text{Log}(4+3i) = \log 5 + i\left[\tan^{-1}\left(\frac{3}{4}\right) + 2n\pi\right]$$

❖ Gregory's Series :

If $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$

This is known as Gregory's series.

Proof :-We have Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \sec\theta e^{i\theta} = \frac{1}{\cos\theta}[\cos\theta + i\sin\theta]$$

$$= 1 + i\tan\theta$$

Taking logarithm on both sides,

$$\log(\sec\theta e^{i\theta}) = \log(1 + i\tan\theta)$$

$$\Rightarrow \log\sec\theta + \log e^{i\theta} = \log(1 + i\tan\theta) \quad \ldots(1)$$

Given that $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$\therefore \tan\theta$ lies between -1 and 1.

And $|i\tan\theta| = |i|.|\tan\theta| \leq 1$

We know that if $|x| \leq 1$,then

$$\log(1+x) = (x) - \frac{(x)^2}{2} + \frac{(x)^3}{3} - \frac{(x)^4}{4} + \cdots$$

$\therefore$For $|i \tan\theta| \leq 1$,

$$\log(1+\mathrm{i}\ \tan\theta) = (\mathrm{i}\ \tan\theta) - \frac{(\mathrm{i}\ \tan\theta)^2}{2} + \frac{(\mathrm{i}\ \tan\theta)^3}{3} - \frac{(\mathrm{i}\ \tan\theta)^4}{4} + \cdots$$

$$\log(1+\mathrm{i}\ \tan\theta) = i\tan\theta + \frac{\tan^2\theta}{2} - i\,\frac{\tan^3\theta}{3} - \frac{\tan^4\theta}{4} + i\,\frac{\tan^5\theta}{5} + \ldots$$

$$\log\sec\theta + i\theta = \left[\frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \cdots\right] + \mathrm{i}\left[\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots\right]$$

Equating real and imaginary parts,

$$\log\sec\theta = \left[\frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \cdots\right]$$

$$\theta = \left[\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots\right] \qquad \ldots(2)$$

Equation (2) is known as Gregory's series.

➢ Generalized form of Gregory's series :

If $n\pi - \frac{\pi}{4} \leq \theta \leq n\pi + \frac{\pi}{4}$ then $\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$

➢ Application of Gregory series :

(1) From Gregory's series:

If $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ then $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$

Take $\tan\theta = x \Rightarrow \theta = \tan^{-1} x$

Then by putting $x = 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(2)Euler's series :

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7}\right) + \cdots$$

Question 1: Prove that $\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right]$

Solution : By Gregory's series, if $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$,

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

Put $\theta = \frac{\pi}{6}$,we get

$$\frac{\pi}{6} = \tan\frac{\pi}{6} - \frac{\tan^3\frac{\pi}{6}}{3} + \frac{\tan^5\frac{\pi}{6}}{5} - \frac{\tan^7\frac{\pi}{6}}{7} + \cdots$$

$$= \frac{1}{\sqrt{3}} - \frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3} + \frac{\left(\frac{1}{\sqrt{3}}\right)^5}{5} - \frac{\left(\frac{1}{\sqrt{3}}\right)^7}{7} + \cdots$$

$$= \frac{1}{\sqrt{3}}\left\{1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right\}$$

$$\Rightarrow \quad \pi = \frac{2\times 3}{\sqrt{3}}\left\{1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right\}$$

$$\therefore \pi = 2\sqrt{3}\left\{1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots\right\}$$