GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

MATHEMATICS

B.Sc. Sem-5

PAPER-1:ANALYSIS

UNIT – III

ANALYTIC FUNCTIONS

SUBJECT: HARMONIC FUNCTIONS, ORTHOGONAL FAMILIES,CONSTRUCTION OF ANALYTIC FUNCTION.

❖ HARMONIC FUNCTIONS :

A real valued function u of two variables x and y is said to be harmonic if it has a continuous partial derivatives and satisfies the equation-

$$u_{xx} + u_{yy} = 0$$

or

$$\nabla^2 u = 0$$

This equation is also known as Laplace equation.

The functions u(x , y) and v(x , y) which satisfy Laplace's equation are called harmonic functions.

➢ Theorem :-

If f(z)=u+iv is an analytic function of z=x+iy then prove that u and v are harmonic functions.

Proof:- Let f(z)=u+iv be an analytic function of z=x+iy.

$\therefore$ By theorem,

C-R equations are satisfied.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ... (1) $\quad$ *and* $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...(2)

<u>To prove :</u>1)u is harmonic.

i.e. $u_{xx} + u_{yy} = 0$

Diff. eqn (1) w.r.t. x and eqn (2) w.r.t. y ,we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

On adding, we get

$$u_{xx} + u_{yy} = 0$$

<u>To prove :</u>2) v is harmonic.

i.e. $v_{xx} + v_{yy} = 0$

Diff. eqn (1) w.r.t. x and eqn (2) w.r.t. y ,we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right)$$

On adding, we get

$$v_{xx} + v_{yy} = 0$$

Hence proved.

❖ ORTHOGONAL FAMILIES :

Two families of curves u(x ,y)=c_1 and v(x ,y)=c_2 are said to be an orthogonal system if they intersect at right angles at each of their points of intersection.

➢ Theorem :-

If w= f(z)=u(x ,y)+i v(x ,y) is an analytic function in domain D, then the one parameter families of curves u(x ,y)=c_1 and v(x ,y)=c_2 form two orthogonal families.

Proof :- Let f(z)=u(x ,y)+i v(x ,y) is an analytic function in domain D.

Hence Cauchy-Riemann equations are satisfied.

i.e. . $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Taking differential of u(x ,y)=c_1 and v(x ,y)=c_2, we get

du =0 and dv = 0

$\Rightarrow \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0 \quad and \quad \frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=0$

$\Rightarrow m_1$ = slope of the tangent to the curve u(x ,y)=c_1 at any point (x ,y)=$\frac{\partial y}{\partial x}=-\frac{u_x}{u_y}$

and m_2 = slope of the tangent to the curve v(x ,y)=c_2 at any point (x ,y)=$\frac{\partial y}{\partial x}=-\frac{v_x}{v_y}$

Now, $m_1m_2=\left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right)=\frac{u_xv_x}{u_yv_y}=\frac{u_xv_x}{(-v_x)u_x}=-1$

$\Rightarrow$The curves u(x ,y)=c_1 and v(x ,y)=c_2 form two orthogonal families.

Question 1:If $u = x^2 - y^2, v = -\frac{y}{x^2+y^2}$ then show that both u and v satisfies Laplace's equation but u+iv is not an analytic function of z.

Proof:-To show that u+iv is not an analytic function of z,we have to show that u and v do not satisfy the Cauchy-Riemann equations $u_x = v_y , u_y = -v_x$.

Now,

$$\frac{\partial u}{\partial x} = 2x , \frac{\partial u}{\partial y} = \text{-2y}$$

$$\frac{\partial v}{\partial x} = -y\left[\frac{(-1)2x}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2} , \frac{\partial v}{\partial y} = \frac{-(x^2+y^2)-2y(-y)}{(x^2+y^2)^2} = -\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

Clearly, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$

⇒ u+iv is not an analytic function of z.

To prove that u and v both satisfy Laplace's equation, we have to show that $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Now, $u_{xx} + u_{yy}$ = 2-2=0

$$\therefore u_{xx} + u_{yy} = 0$$

And

$$v_{xx} + v_{yy} = 2y\left[\frac{\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)2xx}{(x^2+y^2)^4}\right]+\frac{2y\left(x^2+y^2\right)^2-\left(y^2-x^2\right)2\left(x^2+y^2\right)2y}{(x^2+y^2)^4}$$

$$=\frac{2y\left(y^2-3x^2\right)}{(x^2+y^2)^3}+\frac{\left(3x^2-y^2\right)2y}{(x^2+y^2)^3}=0$$

$\therefore v_{xx} + v_{yy}$=0

Hence proved.

❖ Contruction of analytic function :

Method-1 :Milne Thomson's Method:

Case -1]:If the real part u of f(z) is given.

Step-1] Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Step-2] Find f'(z)= $\frac{\partial u}{\partial x}$ +i $\frac{\partial v}{\partial x}$ = $\frac{\partial u}{\partial x}$-i $\frac{\partial u}{\partial y}$

Step-3]Put x=z and y=0 in f'(z).
Step-4]Integrate f'(z) to obtain f(z).

Case -2] If imaginary part v of f(z)is given.

Step-1] Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Step-2] Find f'(z)= $\frac{\partial u}{\partial x}$ +i $\frac{\partial v}{\partial x}$ = $\frac{\partial v}{\partial y}$ +i $\frac{\partial v}{\partial x}$

Step-3] Put x=z and y=0 in f'(z).

Step-4] Integrate f'(z) to obtain f(z).

Method-2:By finding harmonic conjugate v and construction of analytic function f(z) :

Given :-u is given.

Step-1] Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Step-2] dv= $\frac{\partial v}{\partial x}$dx+ $\frac{\partial v}{\partial y}$dy=- $\frac{\partial u}{\partial y}$dx+ $\frac{\partial u}{\partial x}$dy=Mdx + Ndy ...(1)

Step-3] M=- $\frac{\partial u}{\partial y}$, N= $\frac{\partial u}{\partial x}$

Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial y}$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$ then eqn (1) is exact differential.

Step-4]Find v by using,

v= $\int Mdx + \int Ndy + c$

(taking y constant) (terms of N which do not contain x)

Step-5] f(z)=u +iv

Question 1:-Find the analytic function f(z)= u + iv of which the real part is u= $e^x(x\cos y - y\sin y)$.

Solution:-We have u= $e^x(x\cos y - y\sin y)$.

$\frac{\partial u}{\partial x}$ = e^x(cos y)+ e^x(x cosy –ysiny) , $\frac{\partial u}{\partial y}$ = e^x(-xsiny-ycosy-siny)

Method-1]f'(z)= $\frac{\partial u}{\partial x}$ +i $\frac{\partial v}{\partial x}$ = $\frac{\partial u}{\partial x}$-i $\frac{\partial u}{\partial y}$

$\Rightarrow$ f(z)=$\int e^x\, cosy + e^x$(xcosy-ysiny)+ie^x(-xsiny-ycosy-siny)dz

Put x=z and y=0 in above equation,

$\Rightarrow$ f(z)=$\int e^z\, cos0 + e^z$(zcos0-0sin0)+ie^z(-zsin0-0cos0-sin0)dz

=$\int(e^z + e^z$z)dz +c

=e^z - e^z +ze^z +c

f(z) =ze^z+c

Method :-2]dv=$\frac{\partial v}{\partial x}$dx+ $\frac{\partial v}{\partial x}$dy=- $\frac{\partial u}{\partial y}$dx+ $\frac{\partial u}{\partial x}$dy

=-e^x(-xsiny-ycosy-siny)dx +e^x(xcosy-ysiny+cosy)dy

which is of the type dv=Mdx + Ndy,

where M =-e^x(-xsiny-ycosy-siny) and N=e^x(xcosy-ysiny+cosy)

$\frac{\partial M}{\partial y}$=-$e^x$(−xcosy + ysiny − cosy − cosy)

=e^x(2cosy+xcosy-ysiny)

$\frac{\partial N}{\partial y}$=$e^x$cosy+(xcosy-ysiny+cosy) e^x=e^x(2cosy+xcosy-ysiny)

Hence the given eqn is exact differential. Hence its solution is on integrating v ,we get

v=$\int Mdx + \int Ndy + c$

(taking y constant) (terms of N which do not contain x)

v=$\int -e^{x}(-\text{xsiny}-\text{ycosy}-\text{siny})dx + \int 0dy + c$

=siny$\int xe^{x}dx + (ycosy + siny)\int e^{x}dx + 0 + c$

=[(x-1)siny+ycosy+siny] e^{x}+c

=(xsiny+ycosy) e^{x}+c

∴f(z)=u + iv

=$e^{x}(x\cos y - y\sin y) + i$[(xsiny+ycosy) e^{x}+c]

=xe^{x}(cosy +isiny)+iye^{x}(cosy +isiny)+ic

=(x+iy) $e^{x}e^{iy} + c'$

∴f(z)=z$e^{z} + c'$