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Class:- B.Sc.(sem-I)

Subject:-Mathematics

paper II:- Calculus

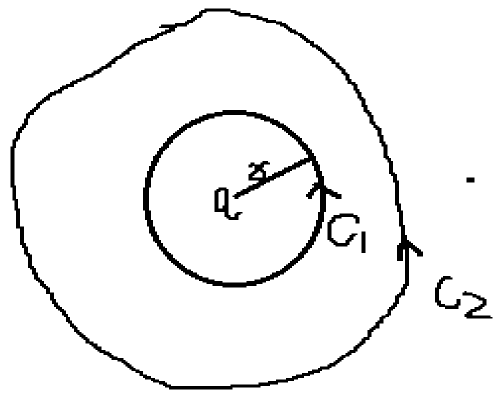
Topic:- Cauchy’s integral Formulae.

Cauchy’s integral Formula

Theorem:- If f(z) be analytic in a simply connected domain D abd let C be a simply closed curve in d oriented counterclockwise. Then for any point a within C,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ or } f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

Proof:-



Let a be any point within a simple closed contour C.

The function $\frac{f(z)}{z-a}$ is not defined at z=a and hence is nor analytic at the point z=a.

In the situation its integral cannot be evaluated by means of

Cauchy's theorem at this point 'a'

We deforms c to a circle C_1 with center 'a'and the radius r so small that C_1 lies wntirely inside C.

Then $\int_C \frac{f(z)}{z-a}$**dz =**$\int_{C_1} \frac{f(z)}{z-a}$**dz ……(1)**

The circle C_1 is given by

|z-a|= r or z-a= r $e^{i\theta}$, 0$\leq \theta \leq 2\pi$, then

$\int_{C_1} \frac{f(z)}{z-a}$**dz =** $\int_0^{2\pi} \frac{f(a+r\,e^{i\theta})}{r\,e^{i\theta}} r\,e^{i\theta}\,i\,d\theta$

$$\int_{C_1} \frac{f(z)}{z-a} dz = i\int_0^{2\pi} f(a + r\, e^{i\theta}) d\theta$$

Let us shrink C_1 to a point a by taking r $\rightarrow 0$.

$$\Rightarrow \lim_{r\to 0} \int_{C_1} \frac{f(z)}{z-a} dz = i \lim_{r\to 0} \int_0^{2\pi} f(a + r\, e^{i\theta}) d\theta$$

$$= i \int_0^{2\pi} \lim_{r\to 0} f(a + r\, e^{i\theta}) d\theta$$

$$= i \int_0^{2\pi} f(a) d\,\theta$$

$$= i\, f(a) \int_0^{2\pi} d\,\theta$$

$$\mathbf{= 2\pi i f(a)}$$

Then (1) $\Rightarrow \lim_{r\to 0} \int_C \frac{\boldsymbol{f(z)}}{\boldsymbol{z-a}} \mathbf{dz} = \lim_{r\to 0} \int_{C_1} \frac{\boldsymbol{f(z)}}{\boldsymbol{z-a}} \mathbf{dz}$

$$\Rightarrow \int_C \frac{\boldsymbol{f(z)}}{\boldsymbol{z-a}} \mathbf{dz} = \lim_{r\to 0} \int_{C_1} \frac{\boldsymbol{f(z)}}{\boldsymbol{z-a}} \mathbf{dz} = \boldsymbol{2\pi i f(a)}$$

- Examples:-Using Cauchy integral formula, evaluate the following integrals.

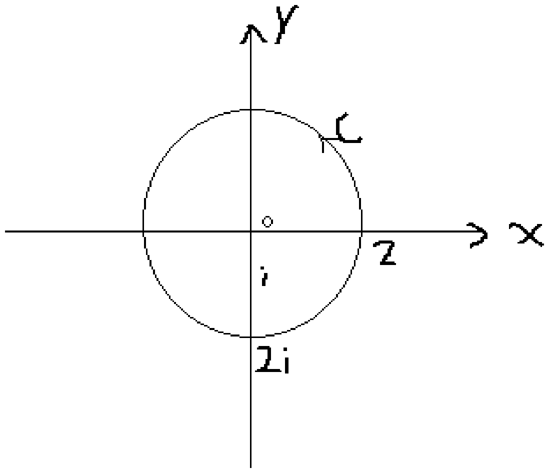
(1) $\int_C \frac{\boldsymbol{z\,dz}}{(\mathbf{9}-z^2)(\boldsymbol{z}-\boldsymbol{i})}$, where C is the circle $|z|$ =2 describe in the positive sense .

Solution:- by Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

$$\Rightarrow \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Where z=a ia a point inside contour C and f(z) is analytic within and upon C.



Let $I = \int_C \frac{\boldsymbol{z\,dz}}{(\mathbf{9}-z^2)(\boldsymbol{z}-\boldsymbol{i})}$

Take f(z) = $\frac{\boldsymbol{z\,dz}}{(\mathbf{9}-z^2)}$ which is analytic within and upon C, since $|z|$=2

Then,

$$I = \int_{\boldsymbol{C}} \frac{\boldsymbol{z\,dz}}{[\boldsymbol{z}-(-\boldsymbol{i})]}$$

$$= 2\pi i f(-i)$$

$$=\left[\frac{-\boldsymbol{i}}{[\mathbf{9}-(-i)^2}\right]$$

$$=\frac{2\pi}{9+1}$$

$$= \frac{\pi}{5}$$

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Thank you