GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

MATHEMATICS

B.Sc. Sem-5

PAPER-1:ANALYSIS

UNIT – III

ANALYTIC FUNCTIONS

SUBJECT: ANALYTIC FUNCTIONS AND CAUCHY-RIEMANN EQUATIONS

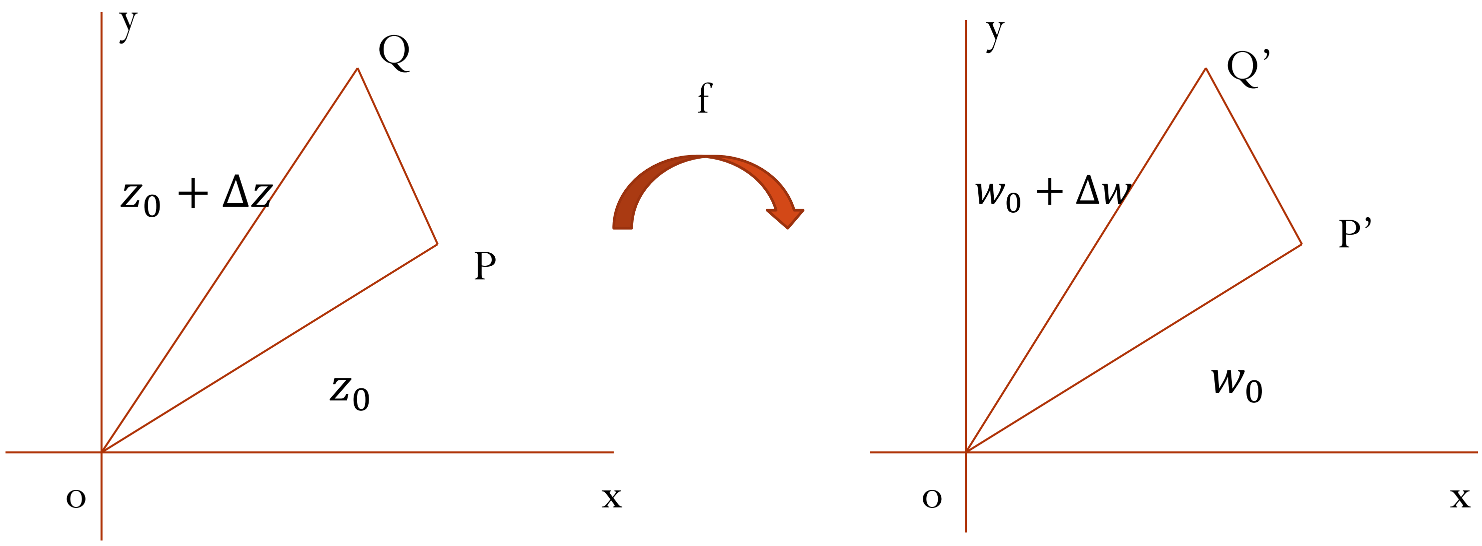
❖ DERIVATIVES :

If f(z) is a single valued function in the z-plane then the derivative of f(z) is defined as-

$$f'(z)= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

provided that limit exist in independent in the way in which Δz converges to zero i.e.$\Delta z \to 0$

Hence f(z) is differenciable.



❖ CAUCHY-RIEMANN EQUATIONS :

If w=f(z)=u(x,y) + iv(x,y) is a single valued function in a domain D then the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called Cauchy-Riemann equations.

Proof :- Let z = x+iy

Then,

$\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial x}$ =i ....(1)

Let f(z)=u+iv ...(2) be analytic in a domain D. ...(given)

∴By defn,

f'(z) exists in D.

Diff. eqn (2) w .r .t. x and y , we get

f'(z) .$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \text{i}\frac{\partial v}{\partial x}$ and f'(z) .$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \text{i}\frac{\partial v}{\partial y}$

But $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = \text{i}$ ...by eqn (1)

$\Rightarrow$f'(z) .1= $\frac{\partial u}{\partial x} + \text{i}\frac{\partial v}{\partial x}$ and f'(z) . i= $\frac{\partial u}{\partial y} + \text{i}\frac{\partial v}{\partial y}$

$$\Rightarrow \frac{1}{i} = \frac{\frac{\partial u}{\partial x} + \text{i}\,\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} + \text{i}\,\frac{\partial v}{\partial y}}$$

$$\Rightarrow \text{i}\left(\frac{\partial u}{\partial x} + \text{i}\,\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + \text{i}\frac{\partial v}{\partial y}$$

$$\Rightarrow \text{i}\frac{\partial u}{\partial x} + i^2\,\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + \text{i}\frac{\partial v}{\partial y}$$

$\Rightarrow$ $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence proved.

❖ POLAR FORM OF CAUCHY-RIEMANN EQUATIONS :

If f(z) = u+iv is an analytic function and z= $re^{i\theta}$,where u,v,r,θ are all real ,show that the Cauchy-Riemann eqn are,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof :- Let z= $re^{i\theta}$

Then,

$$\frac{\partial z}{\partial x} = e^{i\theta} \quad \text{and} \quad \frac{\partial z}{\partial x} = ire^{i\theta} \quad \text{....eqn (1)}$$

Let f(z)=u+iv ...(2) be analytic in a domain D. ...(given)

∴By defn,

f'(z) exists in D.

Diff. eqn (2) w .r .t. x and y , we get

$$f'(z) .\frac{\partial z}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{and} \quad f'(z) .\frac{\partial z}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow f'(z)\,.e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{and} \quad f'(z)\,.ire^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \quad \ldots by\ (1)$$

$$\Rightarrow \frac{1}{ir} = \frac{\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}}{\frac{\partial u}{\partial \theta} + i\,\frac{\partial v}{\partial \theta}}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} + i^2 r\,\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\,\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\,\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence proved.

Question 1 :Prove that the functions $e^x(\cos y + i \sin y)$ is analytic and find its derivative.

Proof :- Let w = u+iv $= e^x(\cos y + i \sin y)$

$\therefore$ u$= e^x \cos y$, v $= e^x \sin y$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = - e^x \sin y = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are satisfied.

Hence w is analytic everywhere.

Now,

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z$$