GOVERNMENT INSTITUTE OF SCIENCE COLLEGE,NAGPUR

# MATHEMATICS

## B.Sc. Sem-5

PAPER-1:ANALYSIS

UNIT – IV

CONFORMAL REPRESENTATION

SUBJECT: MOBIUS TRANSFORMATION,CROSS RATIO,FIXED POINTS,CRITICAL POINTS.

# ❖ MOBIUS TRANSFORMATION OR THE BILINEAR TRANSFORMATION :

The transformation in the form $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ is called a bilinear transformation or Mobius transformation ,where a,b,c,d are the complex constants.

Here $ad - bc \neq 0$ denote the determinant of the transformation.

The transformation $w = \frac{az+b}{cz+d}$ can be expressed as,

$$czw+dw=az+b$$

$$\Rightarrow \quad czw+dw-az-b=0$$

It is linear in bothw and z,so it is bilinear transformation.

The transformation was first studied by the German Geometer A.F. Mobins. (1790-1868)

## ➢ Theorem 1]:-

Prove that the bilinear transformation can be considered as combination of the transformation of translation , rotation , stretching and inversion.

Proof:- Consider the bilinear transformation,

$$w = \frac{az+b}{cz+d}, ad - bc \neq 0$$

$$= \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)}$$

$$= \frac{a\left(z+\frac{d}{c}+\frac{b}{a}-\frac{d}{c}\right)}{c\left(z+\frac{d}{c}\right)}$$

$$= \frac{a}{c}\left[1 + \frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right]$$

$$\Rightarrow \quad w = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$$

Taking $z_1 = z + \frac{d}{c}, z_2 = \frac{1}{z_1}, z_3 = \frac{bc-ad}{c^2} z_2$

We get,

$$w=\frac{a}{c} + z_3$$

The above three transformations namely $z_1, z_2, z_3$ are of the form,

$$w=z+\alpha \,,w=\frac{1}{z} \,, w = \beta z$$

This proves that the every bilinear transformation is the resultant of the bilinear transformations,

$$w=z+\alpha \,, w=\frac{1}{z} \,, w = \beta z$$

where, $w=z+\alpha$ represents translation.

$w=\frac{1}{z}$ represents inversion.

$w = \beta z$ represents rotation and stretching.

Question 1:-Find whether w=$\frac{2z+1}{4z+2}$ is a bilinear transformation.

Solution :-We have,

$$w=\frac{2z+1}{4z+2}$$

Comparing this with he bilinear transformation,

$$w = \frac{az+b}{cz+d}, ad - bc \neq 0$$

Then a=2 ,b=1 ,c=4 ,d=2

Now,

ad-bc=(2)(2)-(1)(4)

= 4-4

=0

Hence w=$\frac{2z+1}{4z+2}$ is not a bilinear transformation.

## ❖ CROSS RATIO :

If $z_1, z_2, z_3, z_4$ are distinct points taken in order in which they are written then the ratio $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}$ is called the cross ratio of $z_1, z_2, z_3, z_4$.
The cross ratio is denoted by $(z_1, z_2, z_3, z_4)$.

$$(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}$$

## ➢ Number of distinct cross ratio :-

Since ther are four letters $z_1, z_2, z_3, z_4$ can be arranged in $4! = 24$ ways, there will be 24 cross ratios.But there will be only six distinct cross ratios. This is because if we interchange any two letters and then interchange two, the cross ratios this letters in this new will be same.

## ➢ Preservarence of cross ratio :-

- Theorem :-

  The cross ratio remains invariant under the bilinear transformation.

Proof:- Let the bilinear map is given by ,

$$w = \frac{az+b}{cz+d}, ad - bc \neq 0 \qquad ...(1)$$

Let $w_1, w_2, w_3, w_4$ be the images of four distinct points $z_1, z_2, z_3, z_4$ in z-plane under a bilinear map then

$$w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(az_1+b)(cz_2+d)-(az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{ad(z_1-z_2)-bc(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \qquad ...(2)$$

Similarly,

$$w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)} \qquad \ldots(3)$$

$$w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)} \qquad \ldots(4)$$

$$w_4 - w_1 = \frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)} \qquad \ldots(5)$$

Then,

$$(w_1, w_2, w_3, w_4) = \frac{(w_4-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w_4)}$$

$$= \frac{(z_4-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z_4)}$$

$$= (z_1, z_2, z_3, z_4)$$

Hence the cross ratio remains invariant underb the bilinear transformation.

Question :-Find the bilinear transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$into the points $w_1 = 1,$

$$w_2 = i w_3 = -1$$

Solution :- The transformation is given by ,

$$(w_1, w_2, w_3, w_4)=(z_1, z_2, z_3, z_4)$$

$$\Rightarrow \frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\Rightarrow \frac{w-1}{w+1} = \left(\frac{z-2}{z+2}\right)\left(\frac{2+i}{2-i}\right)\left(\frac{1-i}{1+i}\right)$$

$$=\left(\frac{z-2}{z+2}\right)\left(\frac{4-1+4i}{4+1}\right)\left(\frac{1-1-2i}{1+1}\right)$$

$$=\left(\frac{z-2}{z+2}\right)\left(\frac{4-3i}{5}\right)$$

By componendo-dividendo,

$$\Rightarrow \frac{w-1+w+1}{w-1-w-1} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

$$\Rightarrow \frac{2w}{-2} = \frac{3z(3-i)+2(1+3i)}{-iz(3-i)-6(3-i)}$$

$$\Rightarrow \frac{w}{-1} = \frac{-(3z+2i)(3-i)}{(iz+6)(3-i)}$$

$$\Rightarrow \text{w}=\frac{3z+2i}{iz+6}$$

This is required bilinear transformation.

## ❖ FIXED POINTS :-

The points which co-incide with their transformations are called the fixed points or invariant points.

i.e. The fixed points of the transformation w=f(z) are obtained from the equation z=f(z).

Consider the bilinear transformation $w = \frac{az+b}{cz+d}$

$$\text{z}= \frac{az+b}{cz+d}$$

$$\Rightarrow cz^2 - (a-d)z - b = 0$$

$$\Rightarrow \text{z}= \frac{(a-d)\pm\sqrt{(a-d)^2-4bc}}{2c}$$

$\Rightarrow z = \frac{(a-d)\pm\sqrt{M}}{2c}$, where $M=(a-d)^2-4bc$

The number of finite fixed points is one or two according as $M=0$ or $M\neq 0$.

- NOTE:-

1. If there are two distinct invariant points p and q then the bilinear transformation may be put in the normal form as $\frac{w-p}{w-q} = k\left(\frac{z-p}{z-q}\right)$
2. If there is only one invariant point p then the bilinear transformation may be put in the normal form as $\frac{1}{w-p} = \frac{1}{z-p} + k$

## ❖ CRITICAL POINTS :-

Consider the bilinear transformation

$$w = T(z) = \frac{az+b}{cz+d} \quad ...(1)$$

$$z = T^{-1} = \frac{b-wd}{wc-a}, \text{if } ad-bc \neq 0 \quad ...(2)$$

The transformation T associates a unique point of the w-plane to any of z-plane except the point $z=\frac{-d}{c}$ when $c \neq 0$ . The trasformation $T^{-1}$ associates a unique point of the z-plane to any of w-plane except the point $w=\frac{a}{c}$ when $c \neq 0$ .

The exceptional points $z=\frac{-d}{c}$ and $w=\frac{a}{c}$ are mapped into the points $w=\infty$ and $z=\infty$ respectively from eqns (1) and (2).

From eqn (1), $\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$

$$\Rightarrow \frac{dw}{dz} = \begin{cases} \infty & \text{, if } z=\frac{-d}{c} \\ 0 & \text{, if } z=\infty \end{cases}$$

The points $z=\frac{-d}{c}$ , $z=\infty$ are called critical points.

- <u>Definations :-</u>

1. A bilinear transformation with one fixed point $z_0$ is called parabolic and is expressed as

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + h \quad \text{,if } z_0 = \infty$$

2. A bilinear transformation with two distinct fixed points $z_1 and\ z_2$ is expressed as

$$\frac{w-z_1}{w-z_2} = k\left(\frac{z-z_1}{z-z_2}\right) \quad \text{,if } k(z- z_1) \neq 0$$

or $$w=z+h \quad \text{,if } w-z_1 = k(z- z_1)$$

A transformation with two different fixed points is called

1)Hyperbolic , if k>0

2)elliptic, if k= $e^{i\alpha}$ and $\alpha \neq 0$.

3)loxodromic, if k=a $e^{i\alpha}$.

where $\alpha \neq 1, \alpha \neq 0$ ; $\alpha$ and a both are real numbers and a>0.

Question:-Find the fixed point and the normal form of the following bilinear transformation. Is any of these transformation hyerbolic, elliptic or parabolic.

1. w=$\frac{z}{z-2}$

Solution:-The fixed point is given by;

$$w=z$$

then,

$$z=\frac{z}{z-2}$$

$$z^2 - 2z - z = 0$$

$$z(z-3)=0$$

$$Z=0\ ,\ 3$$

$$\frac{w-0}{w+3} = k\left(\frac{z-0}{z-3}\right) \quad \text{with } k=-\frac{1}{2}$$

This is the normal form .

Here k$= -\frac{1}{2} = -\frac{1}{2}\, e^{i\pi}$

Hence map is Loxodromic.